$$\begin{bmatrix} U(\tau_1+\tau_2) \\ V(\tau_1+\tau_2) \\ W(\tau_1+\tau_2) \end{bmatrix} = \begin{bmatrix} \cos\Delta\tau_2 & -\sin\Delta\tau_2 & 0 \\ \sin\Delta\tau_2 & \cos\Delta\tau_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\times \begin{bmatrix} \Delta^2\Omega^{-2}\cos\theta_2+1 & -\Delta\Omega^{-1}\sin\theta_2 & \Delta\Omega^{-1}(\cos\theta_2-1) \\ \Delta\Omega^{-1}\sin\theta_2 & \cos\theta_2 & \sin\theta_2 \\ \Delta\Omega^{-1}(\cos\theta_2-1) & -\sin\theta_2 & \Delta^2\Omega^{-2}+\cos\theta_2 \end{bmatrix}$$
$$\times \begin{bmatrix} \cos\Delta\tau_1 & -\sin\Delta\tau_1 & 0 \\ \sin\Delta\tau_1 & \cos\Delta\tau_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\Omega^{-1}(\cos\theta_1-1) \\ \sin\theta_1 \\ \Delta^2\Omega^{-2}+\cos\theta_1 \end{bmatrix}$$

and integrating the resulting $U+\mathrm{i}V$ over the inhomogeneous distribution. The algebra is substantially more complex than that for the two-level evolution matrix technique used in the text, in spite of the simplifying assumption that Ω can be approximated by $\mathcal{R}_0$ [(12.11, 13) do not make this assumption]. This is one more example of how the wave function can involve less algebra in cases with no or uniform decay.

12.4 Show that the differential equations

$$\dot{C}_a = \frac{\mathrm{i}\wp\mathcal{E}(t)}{2\hbar}C_b \ , \qquad \dot{C}_b = \frac{\mathrm{i}\wp\mathcal{E}(t)}{2\hbar}C_a \ ,$$

have the solution

$$C_a(t) = A\cos(\vartheta/2) + B\sin(\vartheta/2)$$

with a corresponding expression for $C_b(t)$, where the partial area

$$\vartheta = \frac{\wp}{\hbar}\int_{-\infty}^{t} \mathrm{d}t'\,\mathcal{E}(t') \ .$$

12.5 Show that the stable solutions of (12.38) are given by $\theta = 2q\pi$, where q is an integer.

13. Field Quantization

Up to now, we have treated many problems in light-matter interactions and have obtained results in excellent agreement with experiments without having to quantize the electromagnetic field. Such a semiclassical description is sufficient to describe most problems in quantum optics. However, there are a few notable exceptions where a classical description of the field leads to the wrong answer. These include spontaneous emission, the Lamb shift, resonance fluorescence, the anomalous gyromagnetic moment of the electron, and "nonclassical" states of light such as squeezed states. The remainder of this book deals with selected problems in light-matter interaction that require field quantization. The present chapter treats the quantization of the electromagnetic field in free space. Those familiar with this subject might want to glance at our notation and then proceed directly to Chap. 14.

Section 13.1 quantizes a single-mode electromagnetic field using the results of Sect. 3.4 for the harmonic oscillator. Section 13.2 generalizes these results to multimode fields. In Sect. 13.3, we find that an electromagnetic field in thermal equilibrium is described by a density matrix leading to a Maxwell-Boltzmann photon statistics. In Sect. 13.4, we review briefly the properties of the coherent states of the electromagnetic field, finding in particular their photon statistics. Section 13.5 discusses the coherence properties of quantum fields in generalization of Sect. 1.4, and Sect. 13.6 discusses the $P(\alpha)$ and other quasi-distributions which allow to cast certain quantum optics problems into a classical-looking formalism. Finally, Sect. 13.7 gives a brief introduction to the second quantization of matter-wave fields.

13.1 Single-Mode Field Quantization

To quantize the electromagnetic field, we consider a cavity of volume V, closed by perfectly reflecting mirrors as diagrammed in Fig. 13.1. For problems in free space, we take this volume to be infinite at the end of the calculation. This needs not be the case and tailored electromagnetic environments can be experimentally realized as discussed in Chap. 18.

A classical monochromatic, single-mode electromagnetic field polarized in the $\hat{x}$-direction has the form

$$\mathbf{E}(z,t) = \hat{x}q(t)[2\Omega^2/\varepsilon_0 V]^{1/2}\sin Kz \ . \tag{13.1}$$