MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Media Laboratory

MAS.961 Quantum Information Science September 27, 2001

Problem Set #2

(due in class, 11-Oct-01)

Instructions: You will be graded only on the *problems* (middle section, below). The *exercises* are for your own enlightenment and practice. Project *questions* need not be handed in; they are candidate questions which you may work on for your final project paper, due at the end of the semester.

Lecture Topics (9/11, 9/13, 9/18, 9/20): Grover algorithms; implementations; open quantum systems

Recommended Reading: Nielsen and Chuang, Chapters 6-8

Exercises:

E1: (**Multiple solution Grover algorithm**) Give explicit steps for the quantum search algorithm for the case of multiple solutions ($1 < M < N/2$).

E2: (**Exact continuous time search algorithm**) Consider simulating the continuous time quantum search algorithm as in Section 6.2 of Nielsen and Chuang. Show that by choosing the timestep Δt appropriately we can obtain a continuous time quantum search algorithm that uses $O(\sqrt{N})$ queries, and for which the final state is $|x\rangle$ *exactly*, that is, the algorithm works with probability 1, rather than with some smaller probability.

E3: (**Eigenstates of photon annihilation**) Prove that a coherent state is an eigenstate of the photon annihilation operator, that is, show $a|\alpha\rangle = \lambda|\alpha\rangle$ for some constant λ.

E4: (**Eigenstates of the Jaynes–Cummings Hamiltonian**) Show that

$$|\chi_n\rangle = \frac{1}{\sqrt{2}}\Big[|n,1\rangle + |n+1,0\rangle\Big] \tag{1}$$

$$|\overline{\chi}_n\rangle = \frac{1}{\sqrt{2}}\Big[|n,1\rangle - |n+1,0\rangle\Big] \tag{2}$$

(where the labels in the ket are $|\text{field},\text{atom}\rangle$) are eigenstates of the Jaynes–Cummings Hamiltonian, $H = \hbar\omega N + \delta Z + g(a^\dagger\sigma_- + a\sigma_+)$ for $\omega = \delta = 0$, with

$$H|\chi_n\rangle = g\sqrt{n+1}|\chi_n\rangle \tag{3}$$

$$H|\overline{\chi}_n\rangle = -g\sqrt{n+1}|\overline{\chi}_n\rangle\,. \tag{4}$$

E5: (**Universality of Heisenberg Hamiltonian**) Show that a swap operation U can be implemented by turning on $J(t)$ for an appropriate amount of time in the Heisenberg coupling Hamiltonian

$$H(t) = J(t)\vec{S}_1\cdot\vec{S}_2 = \frac{J(t)}{4}\Big[X_1X_2 + Y_1Y_2 + Z_1Z_2\Big] \tag{5}$$

to obtain $U = \exp(-i\pi \vec{S}_1 \cdot \vec{S}_2)$. Together with arbitrary single ubit gates, the '$\sqrt{\text{SWAP}}$' gate obtained by turning on the interaction for half this time is universal; compute the $\sqrt{\text{SWAP}}$ and show how to obtain a controlled-NOT gate by composing it with single qubit operations.

E6: (**Measurement**) Suppose we have a single qubit principal system, interacting with a single qubit environment through the transform

$$U = P_0 \otimes I + P_1 \otimes X \,, \tag{6}$$

where X is the usual Pauli matrix (acting on the environment), and $P_0 \equiv |0\rangle\langle 0|, P_1 \equiv |1\rangle\langle 1|$ are projectors (acting on the system). Give the quantum operation for this process, in the operator-sum representation, assuming the environment starts in the state $|0\rangle$.

E7: The graphical method for understanding single qubit quantum operations was derived for trace-preserving quantum operations. Find an explicit example of a non-trace-preserving quantum operation which cannot be described as a deformation of the Bloch sphere, followed by a rotation and a displacement.

E8: ($T_2 \leq T_1$) The T_2 phase coherence relaxation rate is just the exponential decay rate of the off-diagonal elements in the qubit density matrix, while T_1 is the decay rate of the diagonal elements:

$$\begin{bmatrix} a & b \\ b^* & 1-a \end{bmatrix} \rightarrow \begin{bmatrix} (a-a_0)e^{-t/T_1} + a_0 & be^{-t/2T_2} \\ b^* e^{-t/2T_2} & (a_0 - a)e^{-t/T_1} + 1 - a_0 \end{bmatrix} , \tag{7}$$

Amplitude damping has *both* nonzero T_1 and T_2 rates; show that for amplitude damping $T_1 = T_2$. Also show that if amplitude and phase damping are *both* applied then $T_2 \leq T_1$.

E9: (**Exponential sensitivity to phase damping**) Using the Hamiltonian $H = \chi\, a^\dagger a (b + b^\dagger)$ describing the interaction between two harmonic oscillators (a, $a^\dagger$ being the system, and b, $b^\dagger$ being the environment), show that the element $\rho_{nm} = \langle n|\rho|m\rangle$ in the density matrix of the system harmonic oscillator decays exponentially as $e^{-\lambda t (n-m)^2}$ under the effect of phase damping, for some constant λ.

Problems:

P1: (**Lower bound on Grover search**) Suppose the search problem has M solutions. Prove that $O(\sqrt{N/M})$ oracle calls are rqeuired to find a solution using a quantum algorithm.

P2: (**Refocusing dipolar interactions**) The dipolar coupling Hamiltonian is

$$H^{\mathrm{D}}_{1,2} = \frac{\gamma_1 \gamma_2 \hbar}{4r^3} \left[\vec{\sigma}_1 \cdot \vec{\sigma}_2 - 3(\vec{\sigma}_1 \cdot \hat{n})(\vec{\sigma}_2 \cdot \hat{n}) \right] , \tag{8}$$

where γ_1, γ_2, and r are real parameters that describe the physical system, and $\hat{n}$ is a fixed unit vector that can point in any direction. Evolution according to this Hamiltonian for time t gives a unitary operator $U_d(t)$. Our goal is to cause the system to evolve according to the much simpler Hamiltonian

$$H^J_{1,2} = \frac{J\hbar}{4} Z_1 Z_2 \tag{9}$$

for some constant J, using a product of $U_d(t)$ gates and single qubit gates.

(a) Find a series of single-qubit gates to apply that will transform the Hamiltonian as desired. In this part, you may use many gates acting for infinitesimal amounts of time, which allows you to evolve according to a sum of Hamiltonians via the Lie product formula

$$\lim_{n\to\infty}(e^{-iA/n}e^{-iB/n})^n = e^{-i(A+B)}\,. \tag{10}$$

(b) Assume for now that $\hat{n} = \hat{z}$ and find a non-infinitesimal way of simulating $H^J_{1,2}$, i.e., find a finite string of $U_d(t)$ and single-qubit operations whose product is $e^{-it' H^J_{1,2}}$ for some t'.

(c) Generalize your result from the previous part to find a non-infinitesimal way to simulate $H^J_{1,2}$ for arbitary choices of $\hat{n}$.

P3: (**Decoherence free subspaces**) Certain subspaces of Hilbert space remain invariant under decoherence given that the decoherence has certain properties.

(a) Collective phase damping is a quantum operation on two qubits defined by the operators

$$A_0 = \sqrt{p}(IZ + ZI)/2 \quad A_1 = \sqrt{1-p}(IZ + ZI)/2 + (II - ZZ)/2\,, \tag{11}$$

where I and Z denote the identity and Pauli Z operators, respectively, and two such adjacent operators indicate a tensor product, i.e., $IZ = I \otimes Z$. Verify that these operators define a valid quantum operation.

(b) Show that superpositions of the states $|01\rangle$ and $|10\rangle$ do not decohere under collective phase damping. That is, for $|\psi\rangle = a|01\rangle + b|10\rangle$,

$$\mathcal{E}(|\psi\rangle) = \sum_k A_k|\psi\rangle\langle\psi|A_k^\dagger = |\psi\rangle\langle\psi|\,. \tag{12}$$

(c) The quantum operation elements for phase damping of a single qubit are

$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{bmatrix}. \tag{13}$$

Thus independent phase damping is described by the four operators

$$C_0 = \frac{1}{2}B_0 \otimes I\,, \quad C_1 = \frac{1}{2}B_1 \otimes I\,, \quad C_2 = \frac{1}{2}I \otimes B_0\,, \quad C_3 = \frac{1}{2}I \otimes B_1\,. \tag{14}$$

Show that this quantum operation does *not* have a decoherence free subspace. That is, show that there is no two- (or higher) dimensional subspace that is left invariant by independent phase damping.

P4: (**Amplitude damping of a harmonic oscillator**) Suppose that our principal system, a harmonic oscillator, interacts with an environment, modeled as another harmonic oscillator, through the Hamiltonian

$$H = \chi(a^\dagger b + b^\dagger a) \tag{15}$$

where a and b are the annihilation operators for the respective harmonic oscillators.

(a) Using $U = \exp(-iH\Delta t)$, denoting the eigenstates of $b^\dagger b$ as $|k_b\rangle$, and selecting the vacuum state $|0_b\rangle$ as the initial state of the environment, show that the operation elements $E_k = \langle k_b|U|0_b\rangle$ are found to be

$$E_k = \sum_{n=k}^{\infty} \sqrt{\binom{n}{k}} \sqrt{(1-\gamma)^{n-k}\gamma^k}\, |n-k\rangle\langle n| \,, \tag{16}$$

where $\gamma = 1 - \cos^2(\chi\Delta t)$ is the probability of losing a single quantum of energy, and states such as $|n\rangle$ are eigenstates of $a^\dagger a$.

(b) Show explicitly that the operation elements E_k define a trace-preserving quantum operation.

Project Questions:

Q1: (**Continuous time analogues of quantum algorithms**) Discuss continuous-time analogues of one (or more) of the known quantum algorithms. For example, you might consider the Deutsch-Josza problem, Simon's problem, or the quantum Fourier transform.

Q2: (**Thermodynamics of local measurement**) Bennett et al. have conjectured that a set of states can be measured by a local, reversible procedure only if it is *dissectable* (defined in quant-ph/9804053). Prove this conjecture or find a counterexample.