MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Media Laboratory

MAS.961 Quantum Information Science October 11, 2001

Problem Set #3

(due in class, 25-Oct-01)

Instructions: You will be graded only on the *problems* (second section, below). The *exercises* are for your own enlightenment and practice.

Lecture Topics (10/11, 10/16, 10/18, 10/23): distance measures; quantum error correction; q. codes

Recommended Reading: Nielsen and Chuang, Chapters 9-10

Exercises:

E1: What is the trace distance between the probability distribution $(1, 0)$ and the probability distribution $(1/2, 1/2)$? Between $(1/2, 1/3, 1/6)$ and $(3/4, 1/8, 1/8)$?

E2: Show that the trace distance between probability distributions $(p, 1-p)$ and $(q, 1-q)$ is $|p-q|$.

E3: What is the fidelity of the probability distributions $(1, 0)$ and $(1/2, 1/2)$? Of $(1/2, 1/3, 1/6)$ and $(3/4, 1/8, 1/8)$?

E4: (**Existence of fixed points**) *Schauder's fixed point theorem* is a classic result from mathematics that implies that any continuous map on a convex, compact subset of a Hilbert space has a fixed point. Use Schauder's fixed point theorem to prove that any trace-preserving quantum operation $\mathcal{E}$ has a fixed point, that is, ρ such that $\mathcal{E}(\rho) = \rho$.

E5: Suppose $\mathcal{E}$ is a trace-preserving quantum operation for which there exists a density operator ρ_0 and a trace-preserving quantum operation $\mathcal{E}'$ such that

$$\mathcal{E}(\rho) = p\rho_0 + (1-p)\mathcal{E}'(\rho), \tag{1}$$

for some p, $0 < p \leq 1$. Physically, this means that with probability p the input state is thrown out and replaced with the fixed state ρ_0, while with probability $1-p$ the operation $\mathcal{E}'$ occurs. Use joint convexity to show that $\mathcal{E}$ is a strictly contractive quantum operation, and thus has a unique fixed point.

E6: Consider the depolarizing channel, which has operation elements

$$\sqrt{1-\frac{3p}{4}}I \quad , \quad \sqrt{\frac{p}{4}}Y \quad , \quad \sqrt{\frac{p}{4}}X \quad , \quad \sqrt{\frac{p}{4}}Z \tag{2}$$

This process gives $\mathcal{E}(\rho) = pI/2 + (1-p)\rho$. For arbitrary ρ and σ find $D(\mathcal{E}(\rho), \mathcal{E}(\sigma))$ using the Bloch representation, and prove explicitly that the map $\mathcal{E}$ is strictly contractive, that is, $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) < D(\rho, \sigma)$.

E7: (**Concavity of fidelity**) Prove that the fidelity is concave in the first entry,

$$F\left(\sum_i p_i \rho_i, \sigma\right) \geq \sum_i p_i F(\rho_i, \sigma). \tag{3}$$

By symmetry the fidelity is also concave in the second entry.

E8: Show that the minimum fidelity $F(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|))$ when $\mathcal{E}$ is the amplitude damping channel with parameter γ, is $\sqrt{1-\gamma}$.

E9: Show that the syndrome measurement for detecting phase flip errors in the Shor code corresponds to measuring the observables $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$.

E10: Show that recovery from a phase flip on any of the first three qubits of the Shor code may be accomplished by applying the operator $Z_1Z_2Z_3$.

E11: Construct operation elements for a single qubit quantum operation $\mathcal{E}$ that upon input of any state ρ replaces it with the completely randomized state $I/2$. It is amazing that even such noise models as this may be corrected by codes such as the Shor code!

E12: Write an expression for a generator matrix encoding k bits using r repetitions for each bit. This is an $[rk, k]$ linear code, and should have an $rk \times k$ generator matrix.

E13: Explicitly verify that $UX_1U^\dagger = X_1X_2$, $UX_2U^\dagger = X_2$, $UZ_1U^\dagger = Z_1$, and $UZ_2U^\dagger = Z_1Z_2$, where U is the controlled-NOT gate with qubit 1 as the control.

E14: Suppose U and V are unitary operators on two qubits which transform Z_1, Z_2, X_1, and X_2 by conjugation in the same way. Show this implies that $U = V$.

E15: Show that the operations $\bar{Z} = X_1X_2X_3X_4X_5X_6X_7X_8X_9$ and $\bar{X} = Z_1Z_2Z_3Z_4Z_5Z_6Z_7Z_8Z_9$ act as logical Z and X operations on a Shor-code encoded qubit. That is, show that this $\bar{Z}$ is independent of and commutes with the generators of the Shor code, and that $\bar{X}$ is independent of and commutes with the generators of the Shor code, and anti-commutes with $\bar{Z}$.

E16: Give the check matrices for the five and nine qubit codes in standard form.

<u>**Problems:**</u>

P1: (**Distance measures for single qubit operations**) Recall the distance measure

$$E(U, V) = \max_{|\psi\rangle} \|(U - V)|\psi\rangle\|, \tag{4}$$

where the maximum is over all pure states $|\psi\rangle$, and

$$D(U, V) = \mathrm{tr}\left|\sqrt{(U-V)^\dagger(U-V)}\right|. \tag{5}$$

Show that when U and V are single qubit rotations, $U = R_{\hat{m}}(\theta), V = R_{\hat{n}}(\phi)$, $D(U, V) = 2E(U, V)$.

P2: (**Two-bit amplitude damping code**) Amplitude damping is an important process in real physical systems; it models spontaneous emission, inelastic scattering, thermalization of spins to the lattice, and many other microscopic processes where energy is exchanged between the system and environment. In this problem and the next two, we study some quantum codes which correct for this error mechanism. Let $|0_L\rangle = |01\rangle$ and $|1_L\rangle = |10\rangle$ be a quantum code encoding one logical qubit using two physical qubits. Define $|\psi\rangle = a|0_L\rangle + b|1_L\rangle$.

(a) Compute the output state

$$\rho' = \mathcal{E}(|\psi\rangle) = \sum_{j,k=\{0,1\}} (E_j \otimes E_k)|\psi\rangle\langle\psi|(E_j \otimes E_k)^\dagger \tag{6}$$

which results when each physical qubit is subject to amplitude damping, described by the operation elements

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \tag{7}$$

$$E_2 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}. \tag{8}$$

(b) Compute the fidelity $F(|\psi\rangle, \rho') = \sqrt{\langle\psi|\rho'|\psi\rangle}$ of ρ' with respect to $|\psi\rangle$.

(c) Suppose we project the output state into the space orthogonal to $|00\rangle$ (say by performing a measurement of $Z \otimes Z$ to measure the total excitation number, and obtain 0), and keep only the cases when we do not obtain $|00\rangle$. What is this state? What is its fidelity with respect to $|\psi\rangle$?

P3: (**Amplitude damping and the Shor code**) How well does the Shor 9-qubit code correct against amplitude damping errors? Let the operation elements for this process be as above, applied to each physical qubit. Calculate the fidelity of the decoded state as a function of γ. The interesting thing is the power of γ that results; don't worry about getting the pre-factor exactly. You may use Mathematica (or some other computer math package) if you desire, but it is also not complicated to obtain the result by hand.

P4: (**3 qubit amplitude damping code**) Let

Name	Operator						
g_1	X	X	X	X	X	X	X
g_2	Z	Z	Z	Z	I	I	I
g_3	Z	Z	I	I	Z	Z	I
g_4	Z	I	Z	I	Z	I	Z

(9)

be the stabilizer generators for a quantum code.

(a) Give the eight codewords determined by these generators.

(b) Show that this code encodes three qubits and can correct for up to one amplitude damping error on any single qubit.