Lecture # 2, Quantum Computation 2: Cluster model of Q.C.

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Outline:

1. Models of QC
2. One bit teleportation
3. Cluster states
4. Cluster Q.C. model
5. Fault Tolerance
1. MODELS OF Q.C.

Classical models include circuit model, turing model, simulated annealing, etc.

1.1. Quantum Circuit Model

The quantum circuit model includes operations such as $U$, $H$, and $T$. These operations can approximate any circuit and can be combined to form a universal Quantum computation.

1.2. Different Model

This model does not have enough degrees of freedom to create an arbitrary program. No measurement. However, goes pretty far.
1.3. Teleportation Model

\[ |\Psi\rangle \quad \text{B} \quad \text{Fix} \quad \text{U} \quad (\text{stuff}) \quad \text{U}|\Psi\rangle \]

1.4. Cluster Model

Measurement without gates

1. Create State (perhaps hard)

2. Single qubit measurements in weird basis

- feedback from past measurements results used to decide the new measurements

1.5. Resource Comparison

<table>
<thead>
<tr>
<th>Model</th>
<th>States</th>
<th>Gates</th>
<th>Measurements</th>
<th>Fault Tolerant?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q. Circuit</td>
<td>(</td>
<td>0\rangle)</td>
<td>cnot, U</td>
<td>(</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(for FT, only certain logical operations allowed)</td>
</tr>
<tr>
<td>Teleport Q.C.</td>
<td>(</td>
<td>0\rangle,</td>
<td>Prog\rangle)</td>
<td>Clifford + Pauli</td>
</tr>
<tr>
<td>Cluster</td>
<td>(</td>
<td>Cluster\rangle)</td>
<td>None</td>
<td>Arbitrary Single Qubit</td>
</tr>
<tr>
<td>Hamiltonian Q.C.</td>
<td>(</td>
<td>\text{ground}(t)\rangle)</td>
<td>Hamiltonian</td>
<td>(</td>
</tr>
<tr>
<td>(Edward Farhi)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. ONE BIT TELEPORTATION

Creating a Swap Gate

\[
\begin{align*}
|\psi\rangle & \rightarrow \begin{array}{c}
\text{\textcircled{H}} \\
\text{\textcircled{Z}} \\
\text{\textcircled{H}}
\end{array} \\
|\phi\rangle & \rightarrow |\psi\rangle \\
|\phi\rangle & \rightarrow |\psi\rangle \\
|0\rangle & \rightarrow |\psi\rangle
\end{align*}
\]

note that

\[
\begin{align*}
&\text{\textcircled{H}} = \begin{array}{c}
\text{\textcircled{H}} \\
\text{\textcircled{Z}} \\
\text{\textcircled{H}}
\end{array} \\
&\text{\textcircled{Z}} = \begin{array}{c}
\text{\textcircled{Z}} \\
\text{\textcircled{Z}} \\
\text{\textcircled{Z}}
\end{array}
\end{align*}
\]

Takes 11 to -11
Since we don’t care about the $|0\rangle$, 

$$
\begin{array}{c}
|\psi\rangle \\
|0\rangle
\end{array}
\begin{array}{c}
\text{H} \\
\text{H}
\end{array}
\begin{array}{c}
|0\rangle \\
|\psi\rangle
\end{array}
= 
\begin{array}{c}
|\psi\rangle
\end{array}
\begin{array}{c}
\text{H} \\
\text{Z}
\end{array}
\begin{array}{c}
|0\rangle \\
|\psi\rangle
\end{array}
$$

This is Z teleportation. It gives an intuition about what you can do with Q.C. that you can’t do classically.

Another form of teleportation is X teleportation:

$$
\begin{array}{c}
|\psi\rangle \\
|+\rangle
\end{array}
\begin{array}{c}
\text{H} \\
\text{X}
\end{array}
\begin{array}{c}
|+\rangle \\
|\psi\rangle
\end{array}
= 
\begin{array}{c}
|\psi\rangle
\end{array}
\begin{array}{c}
\text{H} \\
\text{X}
\end{array}
\begin{array}{c}
|+\rangle \\
|\psi\rangle
\end{array}
$$

Again, we don’t care about the $|+\rangle$, and this is equivalent to

Another Example:
A lot of odd properties are encapsulated in this circuit (EPR pairs, non locality...)
3. CLUSTER STATES

Definition: A cluster state is qubit CPHASE can have different graph topologies.
To initialize:

1. Init qubits in $|+\rangle$
2. Perform CPHASE between connected nearest neighbors

note: This is a stabilizer state!

3.1. Example 1

Initially the Stabilizer is $\langle XI, IX \rangle$.
At the end of circuit, is $\langle XZ, ZX \rangle$ because

and the Z's pass right through.
3.2. Example 2

Note that CPHASE commutes with CPHASE, so we can place them in any order.

Stabilizer initially is \( \langle \mathbb{I}I, \mathbb{I}X, \mathbb{I}IX \rangle \).
At the end of circuit, is \( \langle \mathbb{X}ZI, \mathbb{Z}XZ, \mathbb{I}ZX \rangle \).
Can write the last stabilizer in a new form:

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Definition: A graph state is a stabilizer state with stabilizer generators \( (\mathbb{I}|A) \), where \( \mathbb{I} \) is the identity, and \( A \) is an adjacency matrix for the graph.
Fact: graph states are a strict subspace of all stabilizer states.
Definition: Local Clifford (LC) ops = \( \langle \mathbb{H}, \mathbb{S} \rangle \)
Fact: All stabilizer states are equivalent to some graph state under local Clifford operations.

3.3. Example 3

Can:

1 Swap Qubits (relabel)

2 Can multiply two rows (add the bitstrings)
3 Can do clifford group operations

take $S = \langle \text{ZZI, IZZ, XXX} \rangle$

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
$$

apply H to bits 2 & 3

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
$$

Multiply stabilizers 2 and 3 (add rows 2 and 3)

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
\end{pmatrix}
$$

Local Clifford Equivalent to

Local Clifford Equivalent to

+ + + + +
3.4. Example 4

$0^5 + 1^5$

$S = \langle X^5, ZZIII, IZZII, IIZZI, IIIZZ \rangle$

H on quibits 2 through 5.

$$
\begin{pmatrix}
X & Z & Z & Z & Z \\
Z & X & I & I & I \\
I & X & X & I & I \\
I & I & X & X & I \\
I & I & I & I & X
\end{pmatrix}
$$

Then, add rows, to get to I form:

$$
\begin{pmatrix}
X & Z & Z & Z & Z \\
Z & X & I & I & I \\
Z & I & X & I & I \\
Z & I & I & X & I \\
Z & I & I & I & X
\end{pmatrix}
$$
3.5. Example: The 5 qubit code

\[
S = \langle \\
XZZXI, (1) \\
ZZXIX, (2) \\
ZXIXZ, (3) \\
XIXZZ, (4) \\
XXXXX (5) \rangle \\
\]

\[
\begin{align*}
(523) & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\
\end{align*}
\]

every graph you draw will be a code (though, what is will correct for is a different matter).
3.6. Example: 7 qubit code

[[7,1,3]]
Clifford group ops and multiplying leads to

There may exist more than 1 graph for a code. We would like a canonical graph.
Challenge: Define a canonical (single) graph state equivalent to a given stabilizer state.
4. CLUSTER MODEL OF QUANTUM COMPUTATION

nielsen: quant-ph/0504097 paper
measurement basis:

Output:

1 Final unprocessed qubits

2 Measurement Record (usually tells about errors. Only collapse useful quantum state at the end.)
4.1. Example 1

Recall that

\[ HZ_\alpha = R_z(\alpha) \equiv e^{-\frac{\alpha}{2}} \]

A simple example:

\[ |+\rangle \quad \text{HZ}_\alpha \quad \text{HZ}_\theta \quad \text{HZ}_\phi \quad |+\rangle \]

Recall \( XZ_\theta X = Z_{-\theta} \).

So, choose the \( \pm \) sign dependent on the measurement \( m_1 \).

- \( m_1 = 1 \rightarrow + \) sign
- \( m_1 = 0 \rightarrow - \) sign

This is just a fix up operator.

\[ X^{m_2}HZ_{\pm\phi}X^{m_1}HZ_\theta |+\rangle \rightarrow \]

\[ X^{m_2}HZ_{(-1)^{m_1}\phi}X^{m_1}HZ_\theta |+\rangle = \]

\[ X^{m_2}HX^{m_1}Z_\phi HZ_\theta |+\rangle = \]
While we don’t want the $X^{m_2}Z^{m_1}HZ_\phi HZ_\theta|+\rangle$ portion of above, we know $m_1$ and $m_2$, so we can do teleportation to remove them (Ex. $HZH$ to get an $X$). The $HZ_\phi HZ_\theta|+\rangle$ is what we wanted.

“Pauli Frame” Q. C. qubit → qubit + 2 classical bits to keep track of possible $X$ and/or $Z$ errors.

4.2. Example 2

\[
\begin{align*}
X^{m_2}Z^{m_1}HZ_\phi HZ_\theta|+\rangle
\end{align*}
\]

While we don’t want the $X^{m_2}Z^{m_1}$ portion of above, we know $m_1$ and $m_2$, so we can do teleportation to remove them (Ex. $HZH$ to get an $X$). The $HZ_\phi HZ_\theta|+\rangle$ is what we wanted.

“Pauli Frame” Q. C. qubit → qubit + 2 classical bits to keep track of possible $X$ and/or $Z$ errors.
\[ |\alpha\rangle \xrightarrow{XH} |\beta\rangle \xrightarrow{XH} |\gamma\rangle \xrightarrow{XH} |\alpha\rangle \xrightarrow{H} X \]

\[ |\beta\rangle \xrightarrow{Z} |\alpha\rangle \xrightarrow{H} X \xrightarrow{H} X \]

\[ |\beta\rangle \xrightarrow{Z} |\alpha\rangle \xrightarrow{H} H \xrightarrow{H} X \]

\[ |\beta\rangle \xrightarrow{XZ} |\alpha\rangle \xrightarrow{H} H \xrightarrow{XZ} \text{CNOT pauli frame} \]